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RESEARCH ARTICLE

From massive gravity to modified general relativity II

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Abstract We continue our investigation of massive gravity in the massless limit of vanishing graviton mass. From gauge invariance we derive the most general coupling between scalar matter and gravity. We get further couplings beside the standard coupling to the energy–momentum tensor. On the classical level this leads to a further modification of general relativity.

Keywords Quantum gauge theory · Quantum gravity

1 Introduction

The title "From ... to ..." indicates that our starting point and our very basis is massive gravity. This theory was constructed in previous papers as the spin-2 quantum gauge theory with a graviton mass $m > 0$ (see in particular the new edition of [9]). Since the graviton mass is unmeasurably small we consider the limit $m \to 0$. The essential point is that this does not give the standard mass zero gauge theory with the usual two physical degrees of freedom for the graviton. As a relic of the massive theory the vector graviton field v^μ (which is now also massless) remains as a dynamical actor because it does not decouple from the symmetric tensor field $h^{\mu\nu}$. In the previous paper [8] the influence of this additional coupling was studied. However, this theory is not yet

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complete because the coupling to normal matter was described in the standard way by means of the energy–momentum tensor of ordinary matter. Then there is no direct coupling between the v-field and ordinary matter. But it is our ultimate aim to derive all couplings from gauge invariance. For the scalar matter couplings this is done in this paper. We will find that further couplings between the scalar field Φ and the vector graviton field v^μ with arbitrary coupling constant are possible in massive gravity. In order to get a non-trivial limit $m \to 0$ we have to choose the free coupling constants proportional to the graviton mass m. This is not unusual because coupling terms with factors m appear also at other places in massive gravity. Then in the classical limit the new couplings lead to additional terms in the classical Lagrangean. Those terms on the one hand modify the energy–momentum tensor in Einstein's equations, on the other hand there are further modifications in the field equations for the v-field and the matter field. So our modification of general relativity is not postulated but derived from a strong formulation of gauge invariance in the massless limit of massive spin-2 theory. It may well be that this is more fundamental than arguing in the framework of classical Lagrangean field theory which is the usual method to get modifications of general relativity.

The paper is organized as follows. In Sect. 2 we derive the most general gauge invariant trilinear coupling between scalar matter and massive gravity. We apply the descent method which was already used for the construction of pure massive gravity [6]. To get uniqueness of the result the cohomological methods developed in [3] and [4] have to be employed. We find five possible trilinear couplings where three of them contain the vector graviton field v^μ.

In Sect. 3 we study second order gauge invariance which gives the quartic couplings of the theory. We also get further restrictions on the trilinear couplings. Only three coupling terms survive: one is the well-known coupling to the energy–momentum tensor of the scalar field Φ, the second is the Φ^3 self-coupling and there is one coupling to the v-field. However, the necessary finite renormalizations generate the quartic couplings as always in causal gauge theory.

In Sect. 4 we investigate the new couplings in the limit of vanishing graviton mass $m \to 0$. Since two of the quartic couplings contain m in the denominator, a non-trivial limit only exists if the (free) coupling constants are proportional to m. The new coupling terms then give rise to the modification of general relativity mentioned above.

2 Gauge invariant couplings to scalar matter

In [5] we have analyzed the interaction of massless gravity with massive Yang–Mills fields and with scalar fields. The coupling of the free quantum fields can be obtained with the cohomology methods developed in [3] and [4]. The case of massive gravity can be analyzed with the same methods. We work in the same setting as in [8]. Our fundamental free fields are a symmetric tensor field $h^{\mu\nu}$ which is related to Einstein's $g^{\mu\nu}$ and, in the massive case, the vector-graviton field v^μ together with the related ghost and anti-ghost fields u^μ, $\tilde{u}^\mu$. In addition we consider a scalar "matter" field Φ. The gauge structure on these free asymptotic fields is defined by the gauge charge operator Q and the corresponding gauge variation d_Q. First we give the expression of the gauge invariant variables. It is convenient to introduce the following notations:

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} \qquad \hat{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu}\, h \tag{2.1}$$

and the we define the *Christoffel symbols* according to:

$$\Gamma_{\mu;\nu\rho} \equiv \partial_\rho \hat{h}_{\mu\nu} + \partial_\nu \hat{h}_{\mu\rho} - \partial_\mu \hat{h}_{\nu\rho}. \tag{2.2}$$

The expression

$$R_{\mu\nu;\rho\sigma} \equiv \partial_\rho \Gamma_{\mu;\nu\sigma} - (\rho \leftrightarrow \sigma) \tag{2.3}$$

is called the *Riemann tensor* and it is gauge invariant for massless and massive gravity also. In the case of massive gravity we have new gauge invariants namely the (symmetric) tensor

$$\phi_{\mu\nu} \equiv -\partial_\mu v_\nu - \partial_\nu v_\mu + \eta_{\mu\nu} \partial_\rho v^\rho + m\, h_{\mu\nu} \tag{2.4}$$

and its trace:

$$\phi \equiv \eta^{\mu\nu}\, \phi_{\mu\nu}. \tag{2.5}$$

These expression are immediately proved to be gauge invariant. The same is true for their derivatives and the traceless part of these tensors. Let us denote by $R^{(0)}_{\mu\nu;\rho\sigma;\lambda_1,\ldots,\lambda_n}, \phi^{(0)}_{\mu\nu;\rho_1\ldots\rho_n}, \phi^{(0)}_{;\rho_1\ldots\rho_n}$ the traceless parts of these tensors. The co-cycles of d_Q are the Wick polynomials p satisfying $d_Q p = 0$. We denote the co-cycles by Z_Q. Then we have the following result [4]:

Theorem 2.1 *Let $p \in Z_Q$. Then p is cohomologous to a polynomial in the traceless variables described above.*

We note that in the case of null mass the operator $d_Q$raises the canonical dimension by one unit and this fact is not true anymore in the massive case. We are lead to another cohomology group. Let us take as the space of co-chains the space $\mathcal{P}^{(n)}$ of polynomials of canonical dimension $\omega \leq n$; then $Z^{(n)}_Q \subset \mathcal{P}^{(n)}$ and $B^{(n)}_Q \equiv d_Q \mathcal{P}^{(n-1)}$ are the co-cycles and the co-boundaries respectively. It is possible that a polynomial is a co-boundary as an element of $\mathcal{P}$ but not as an element of $\mathcal{P}^{(n)}$. The situation is described by the following generalization of the preceding theorem.

Theorem 2.2 *Let $p \in Z^{(n)}_Q$. Then p is cohomologous to a polynomial of the form $p_1 + d_Q p_2$ where $p_1 \in \mathcal{P}_0$ and $p_2 \in \mathcal{P}^{(n)}$.*

We will call the co-cycles of the type p_1(resp. $d_Q p_2$) *primary* (resp. *secondary*). Using this result one can determine the most general form of the interaction between the massive gravity and a scalar field of mass M. We will call expressions of the type $d_Q B^I + i\partial_\mu b^{I\mu}$ *relative coboundaries.*

Theorem 2.3 *Suppose that the interaction Lagrangean T between the massive gravity and a scalar field Φ is trilinear in the fields (and their derivatives). Then T it is relatively cohomologous to the following expression:*

$$T = c_1 \Phi \phi_{\mu\nu} \phi^{\mu\nu} + c_2 \Phi \phi^2 \ + c_3 \Phi^2 \ \phi + c_4 \left(\partial_\mu \Phi \partial_\nu \Phi h^{\mu\nu} - \frac{1}{2} M^2 \Phi^2 h \right) + c_5 \Phi^3 \tag{2.6}$$

i.e.

$$d_Q T = i \ \partial_\mu T_0^\mu \tag{2.7}$$

with

$$T_0^\mu = c_4 \left(\frac{1}{2} u^\mu \partial^\nu \Phi \partial_\nu \Phi - u^\nu \partial^\mu \Phi \partial_\nu \Phi - \frac{1}{2} M^2 u^\mu \Phi^2 \right). \tag{2.8}$$

Proof (i) By hypothesis we have (2.7) and the descent procedure (based on a variant of the Poincaré lemma [3]) leads to

$$\begin{aligned} d_Q T^\mu &= i \ \partial_\nu T^{[\mu\nu]}. \\ d_Q T^{[\mu\nu]} &= i \ \partial_\rho T^{[\mu\nu\rho]} \\ d_Q T^{[\mu\nu\rho]} &= i \ \partial_\sigma T^{[\mu\nu\rho\sigma]} \\ d_Q T^{[\mu\nu\rho\sigma]} &= 0 \end{aligned} \tag{2.9}$$

where the carets indicate antisymmetry and can choose the expressions T^I to be Lorentz covariant; we also have

$$gh(T^I) = |I|, \quad \omega(T^I) \leq 5. \tag{2.10}$$

From the last relation in (2.9) we find, using the preceding Theorem 2.2, that

$$T^{[\mu\nu\rho\sigma]} = d_Q B^{[\mu\nu\rho\sigma]} + T_0^{[\mu\nu\rho\sigma]} \tag{2.11}$$

with $T_0^{[\mu\nu\rho\sigma]} \in \mathcal{P}_0^{(5)}$ depending only on the invariants. It is easy to prove that such a (trilinear) expression does not exists so we have

$$T^{[\mu\nu\rho\sigma]} = d_Q B^{\mu\nu\rho\sigma}. \tag{2.12}$$

The third relation of the descent equations gives:

$$d_Q (T^{[\mu\nu\rho]} - i \ \partial_\sigma B^{[\mu\nu\rho\sigma]}) = 0 \tag{2.13}$$

so we obtain again with the preceding Theorem

$$T^{[\mu\nu\rho]} = B^{[\mu\nu\rho]} + i\ \partial_\sigma B^{[\mu\nu\rho\sigma]} + T_0^{[\mu\nu\rho]} \tag{2.14}$$

where $T_0^{[\mu\nu\rho]} \in \mathcal{P}_0^{(5)}$ depends only on the invariants. Again, we can see that such an expression does not exists so we have

$$T^{[\mu\nu\rho]} = B^{[\mu\nu\rho]} + i\ d_\sigma B^{[\mu\nu\rho\sigma]}. \tag{2.15}$$

The second descent equation then gives

$$d_Q(T^{[\mu\nu]} - i\ d_\rho B^{[\mu\nu\rho]}) = 0. \tag{2.16}$$

(ii) We obtain from the relation (2.16) with the preceding Theorem 2.2

$$T^{[\mu\nu]} = d_Q B^{[\mu\nu]} + i\ d_\rho B^{[\mu\nu\rho]} + T_0^{[\mu\nu]} \tag{2.17}$$

where $T_0^{[\mu\nu]} \in \mathcal{P}_0^{(5)}$. The first descent equation gives the restriction:

$$d_Q(T^\mu - \partial_\rho B^{[\mu\nu]}) = \partial_\nu T_0^{[\mu\nu]} \tag{2.18}$$

so the divergence $\partial_\nu T_0^{[\mu\nu]}$ must be a coboundary. We do have a nontrivial expression for $T_0^{[\mu\nu]}$ given by secondary cocycles. In the even sector with respect to parity we have

$$\begin{aligned} T_0^{[\mu\nu]} = {} & g_1 u^\mu u^\nu \Phi + g_2 u^{[\mu\rho]} u^{[\nu\sigma]} \eta_{\rho\sigma} \Phi + g_3 u^{[\mu\nu]} u^\rho \partial_\rho \Phi + g_4 (u^{[\mu\rho]} u^\nu - u^{[\nu\rho]} u^\mu) \partial_\rho \Phi \\ & + g_5 (u^{[\mu\rho]} \partial^\nu \Phi - u^{[\nu\rho]} \partial^\mu \Phi) u_\rho + g_6 (u^\mu u_\rho \partial^\nu \partial^{\rho\Phi} - u^\nu u_\rho \partial^\mu \partial^\rho \Phi) \end{aligned} \tag{2.19}$$

and in the odd sector we have the expression $\epsilon^{\mu\nu\rho\sigma} T'_{[\rho\sigma]}$ where $T'_{[\rho\sigma]}$ has the same form as above but with $g_j \to g'_j$. Here we have used the following notation:

$$u^{[\mu\nu]} = \partial^\mu u^\nu - \partial^\nu u^\mu \tag{2.20}$$

One computes the divergence $\partial_\nu T_0^{[\mu\nu]}$ and requires that it is a coboundary. After some computations one finds out that the remaining terms can be grouped into a relative coboundary i.e. $T_0^{[\mu\nu]} = d_Q b^{\mu\nu} - i \partial_\rho b^{[\mu\nu\rho]}$. It follows that we have

$$T^{[\mu\nu]} = d_Q B^{[\mu\nu]} + i\ d_\rho B^{[\mu\nu\rho]} \tag{2.21}$$

if we redefine the expressions $B^{[\mu\nu]}$ and $B^{[\mu\nu\rho]}$.

The first descent equation gives

$$d_Q(T^\mu - i \partial_\rho B^{[\mu\nu]}) = 0 \tag{2.22}$$

so if we use the Theorem 2.2 we find

$$T^{\mu} = d_Q B^{\mu} + i\partial_{\rho} B^{[\mu\nu]} + T_0^{\mu} \tag{2.23}$$

where $T_0^{\mu} \in \mathcal{P}_0^{(5)}$. If we substitute this in the starting relation (2.7) we get the consistency condition

$$d_Q(T^{\mu} - i\partial_{\mu} B^{\mu}) = i\partial_{\mu} T_0^{\mu} \tag{2.24}$$

i.e. the divergence $\partial_{\mu} T_0^{\mu}$ must be a coboundary. The generic form of T_0^{μ} is again a secondary cocycle. In the even sector with respect to parity we have:

$$T_0^{\mu} = f_1 u^{\mu} \Phi^2 + f_2 u^{\mu} \partial^{\nu} \Phi \partial_{\nu} \Phi + f_3 u^{\nu} \partial^{\mu} \Phi \partial_{\nu} \Phi + f_4 u^{[\mu\nu]} \Phi \partial_{\nu} \Phi + f_5 u_{\nu} \Phi \partial^{\mu} \partial^{\nu} \Phi. \tag{2.25}$$

In the odd sector we have

$$T_0^{\mu} = f' \epsilon^{\mu\nu\rho\sigma} u_{[\nu\rho]} \Phi \partial_{\sigma} \Phi. \tag{2.26}$$

We compute the divergence $\partial_{\mu} T_0^{\mu}$ and the consistency condition leads to

$$T_0^{\mu} = f\left(\frac{1}{2} u^{\mu} \partial^{\nu} \Phi \partial_{\nu} \Phi - u^{\nu} \partial^{\mu} \Phi \partial_{\nu} \Phi - \frac{1}{2} M^2 u^{\mu} \Phi^2\right) + d_Q b_0^{\mu} + \partial_{\nu} b_0^{\mu\nu} \tag{2.27}$$

for some arbitrary constant f. One can get rid of the relative coboundary by redefining the expressions B^{μ} and $B^{\mu\nu}$. Moreover one proves that $\partial_{\mu} T_0^{\mu} = -i d_Q t$ where

$$t \equiv f\left(h_{\mu\nu} \partial^{\mu} \Phi \partial^{\nu} \Phi - \frac{1}{2} M^2 h \Phi^2\right) \tag{2.28}$$

The starting relation (2.7) is now

$$d_Q(T - t - i\partial_{\mu} B^{\mu}) = 0 \tag{2.29}$$

so that a final use of the Theorem 2.2 gives

$$T = t + d_Q B + i\partial_{\mu} B^{\mu} + T_0 \tag{2.30}$$

with $T_0 \in \mathcal{P}_0^{(5)}$. The generic form of T_0 is

$$T_0 = c_1 \Phi \phi_{\mu\nu}^{(0)} \phi^{(0)\mu\nu} + c_2 \Phi \phi^2 + c_3 \Phi^2 \phi + c_4 \Phi^3 \tag{2.31}$$

The expression from the statement follows easily: we can replace $\phi_{\mu\nu}^{(0)}$ by $\phi_{\mu\nu}$ if we redefine the constant c_2 and T^{μ} follows from (2.27). □

3 Second order gauge invariance

In second order we must construct chronological products $T(x, y)$ and $T_\mu(x, y)$ such that

$$d_Q T(x, y) = i \frac{\partial}{\partial x^\mu} T^\mu(x, y) + x \leftrightarrow y \tag{3.1}$$

is verified. The construction procedure is well-known: one first computes the causal commutators $[T(x), T(y)]$ and $[T_\mu(x), T(y)]$ and substitutes the causal Pauli-Jordan distributions in the tree graph contributions by Feynman propagators $D^F(x - y)$. If on the right-hand side of (3.1) a wave operator ∂^2 operates on D^F we obtain a local term $\sim \delta(x - y)$. These anomalies must be compensated by finite renormalizations.

The generic form of the anomaly is

$$A(x, y) = \delta(x - y)a(x) + [\partial^x_\mu \delta(x - y)]a^\mu(x, y) \tag{3.2}$$

The total anomaly is obtained by adding the contribution $A(y, x)$ with x, y interchanged. Then the terms with $\partial\delta$ can be combined by means of the identity

$$[\partial^x_\mu \delta(x - y)]f(x, y) + x \leftrightarrow y = [\partial^y_\mu f - \partial^x_\mu f]\delta(y - x), \tag{3.3}$$

which follows by smearing with symmetric test functions; this is the right test function space here, due to the symmetry of the chronological products. Then the total anomaly is equal to

$$A_{\rm tot}(x, y) = [2a(x) + \partial^y_\mu a^\mu - \partial^x_\mu a^\mu]\delta(x - y) \equiv A(x)\delta(x - y). \tag{3.4}$$

The cancellation of the anomalies is equivalent to

$$A_{\rm tot}(x, y) = d_Q R(x, y) - i\partial_\mu R^\mu(x, y) + x \leftrightarrow y; \tag{3.5}$$

here the expressions $R(x, y)$ and $R^\mu(x, y)$ are *finite renormalizations*: these are quasi-local operators:

$$R(x, y) = \delta(x - y)B(x) + \cdots \tag{3.6}$$

and

$$R^\mu(x, y) = \delta(x - y)B^\mu(x) + \cdots \tag{3.7}$$

where B and B^μ are some Wick polynomials and $\cdots$ are similar terms with derivatives on the delta distribution. Indeed, in this case one can eliminate the anomaly by redefinition of the chronological products

$$T(x, y) \to T(x, y) + R(x, y) \tag{3.8}$$

and

$$T^{\mu}(x,y) \to T^{\mu}(x,y) + R^{\mu}(x,y). \tag{3.9}$$

One can prove that the cancellation (3.5) of the anomalies is achieved if we can write the operator part $A(x)$ in (3.4) in the form

$$A = d_Q B - i\partial_\mu B^\mu. \tag{3.10}$$

In fact, the derivative terms in (3.2) can be combined with help of the identity

$$\partial_\mu^x [B^\mu(x,y)\delta(x-y)] + x \leftrightarrow y = [\partial_\mu^x B^\mu + \partial_\mu^y B^\mu]\delta(x-y). \tag{3.11}$$

The terms in T_μ which generate anomalies are the following ones:

$$\begin{aligned} T_\mu^{an} &= u^\alpha(2\partial_\alpha h^{\varrho\nu}\partial_\mu h_{\varrho\nu} - \partial_\alpha h\partial_\mu h + 2\partial_\alpha u^\nu \partial_\mu \tilde{u}_\nu - 2\partial_\alpha\partial_\nu u^\nu \tilde{u}_\mu) \\ &\quad +2\partial_\nu u^\nu h^{\alpha\varrho}\partial_\mu h_{\alpha\varrho} - \partial_\nu u^\nu h\partial_\mu h + 2\partial_\nu u_\alpha h^{\alpha\nu}\partial_\mu h - 4\partial^\nu u_\alpha \partial_\mu h^{\alpha\varrho} h_{\nu\varrho} \\ &\quad -4u^\alpha \partial_\alpha v^\nu \partial_\mu v_\nu - c_4 u^\alpha \partial_\alpha \Phi \partial_\mu \Phi. \end{aligned} \tag{3.12}$$

Here we have put the gravitational coupling constant $\kappa = 1$ for simplicity. According to Theorem 2.3 the first order coupling to the scalar field Φ of mass M is given by

$$\begin{aligned} T_\Phi &= c_1 \Phi \phi_{\mu\nu}\phi^{\mu\nu} + c_2 \Phi(mh + 2\partial_\mu v^\mu)^2 + c_3 \Phi^2(mh + 2\partial_\mu v^\mu) \\ &\quad + c_4\left(\partial_\mu\Phi\partial_\nu\Phi h^{\mu\nu} - \frac{1}{2}M^2\Phi^2 h\right) + c_5\Phi^3. \end{aligned} \tag{3.13}$$

We first consider the couplings linear in Φ, i.e. with coefficients c_1, c_2.

Theorem 3.1 *Second order gauge invariance implies* $c_1 = 0$ *and* $c_2 = 0$.

Proof To prove this result it is sufficient to find anomalies with c_1 or c_2, which cannot be compensated. For c_1 we consider the commutator

$$-8c_1 u^\lambda \partial_\lambda v^\nu [\partial_\mu v_\nu(x), \phi^{\alpha\beta}(y)]\phi_{\alpha\beta}(y)\Phi \tag{3.14}$$

As described above the commutator gives a causal propagator which in the chronological product becomes a Feynman propagator. Applying the derivative $\partial/\partial x^\mu$ we get a $\partial^2 D^F$ leading to the anomaly

$$A_1 = 4ic_1 u^\lambda \partial_\lambda v^\nu(x)\Phi(y)\left(2\phi_{\alpha\nu}(y)\partial_y^\alpha - \phi(y)\partial_\nu^y\right)\delta(x-y). \tag{3.15}$$

In the same way we consider the commutator

$$-8c_2 u^\alpha \partial_\alpha v^\nu [\partial_\mu v_\nu(x), \phi(y)]\phi(y)\Phi(y). \tag{3.16}$$

Here the resulting anomaly is equal to

$$A_2 = -8ic_2 u^\alpha \partial_\alpha v^\nu(x)\Phi(y)\phi(y)\partial_\nu^y \delta(x-y). \tag{3.17}$$

There are no other anomalies with Wick monomials $uv\phi\Phi$, $uv\phi_{\mu\nu}\Phi$, respectively. Consequently, A_1 and A_2 must cancel against each other. For the last Wick monomial we see from (3.15) that c_1 must be 0 and hence, c_2 must also vanish.

The situation is non-trivial for the remaining couplings which are bilinear and trilinear in Φ. □

Theorem 3.2 *Second order gauge invariance implies $c_4 = -2$, but c_3 and c_5 remain unrestricted. In the second-order chronological products the following finite renormalizations are necessary*

$$T(x,y) = T^F(x,y) + i\delta(x-y)N(x) \qquad T_\mu(x,y) = T_\mu^F(x,y) + i\delta(x-y)N^\mu(x) \tag{3.18}$$

where

$$N = 2\Phi^2 \left\{ M^2(2h^{\mu\nu}h_{\mu\nu} - h^2) + c_3\left[m(2h^{\mu\nu}h_{\mu\nu} - h^2) + \frac{8}{m}(\partial_\mu v^\mu \partial_\nu v^\nu - \partial_\mu v^\nu \partial_\nu v^\mu)\right] \right.$$
$$\left. -\frac{12}{m}c_5 v^\mu \partial_\mu \Phi \right\} \tag{3.19}$$

and

$$N^\mu = 8(u^\mu h^{\alpha\beta} - u^\beta h^{\alpha\mu})\partial_\alpha\Phi\partial_\beta\Phi - (2M^2 + 2mc_3)u^\mu h\Phi^2$$
$$-2c_3(2u^\mu\partial_\alpha v^\alpha - u^\alpha\partial_\alpha v^\mu)\Phi^2. \tag{3.20}$$

Proof In this proof we must calculate all anomalies containing Φ. We also give the commutators where the anomalies come from. From

$$(-u^\alpha\partial_\alpha h - \partial_\alpha u^\alpha h + 2\partial^\nu u^\alpha h_{\alpha\nu})[\partial_\mu h(x), h(y)]\left(mc_3 - \frac{M^2}{2}c_4\right)\Phi^2$$

we get the anomaly

$$A_1 = 2i(2mc_3 - M^2 c_4)(-u^\alpha\partial_\alpha h - \partial_\alpha u^\alpha h + 2\partial^\nu u^\alpha h_{\alpha\nu})\Phi^2\delta, \tag{3.21}$$

and

$$2(u^\lambda\partial_\lambda h^{\alpha\nu} + \partial_\lambda u^\lambda h^{\alpha\nu} - \partial_\lambda u^\alpha h^{\lambda\nu} - \partial_\lambda u^\nu h^{\alpha\lambda})[\partial_\mu h_{\alpha\nu}(x), h(y)]\left(mc_3 - \frac{M^2}{2}c_4\right)\Phi^2$$

leads to

$$A_2 = i(2mc_3 - M^2c_4)(u^\lambda\partial_\lambda h + \partial_\lambda u^\lambda h - 2\partial_\lambda u^\alpha h_{\alpha\lambda})\Phi^2\delta, \tag{3.22}$$

The commutator

$$(-u^{\alpha}\partial_{\alpha}h - \partial_{\alpha}u^{\alpha}h + 2\partial^{\nu}u^{\alpha}h_{\alpha\nu})[\partial_{\mu}h(x), h^{\beta\gamma}(y)]c_4\partial_{\beta}\Phi\partial_{\gamma}\Phi$$

gives

$$A_3 = ic_4(-u^{\alpha}\partial_{\alpha}h - \partial_{\alpha}u^{\alpha}h + 2\partial^{\nu}u^{\alpha}h_{\alpha\nu})\partial_{\beta}\Phi\partial^{\beta}\Phi\delta \tag{3.23}$$

and

$$2(u^{\lambda}\partial_{\lambda}h^{\alpha\nu} + \partial_{\lambda}u^{\lambda}h^{\alpha\nu} - \partial_{\lambda}u^{\alpha}h^{\lambda\nu} - \partial_{\lambda}u^{\nu}h^{\alpha\lambda})[\partial_{\mu}h_{\alpha\nu}(x), h^{\beta\gamma}(y)]c_4\partial_{\beta}\Phi\partial_{\gamma}\Phi$$

yields

$$\begin{aligned} A_4 = -ic_4[&2u^{\lambda}\partial_{\lambda}h^{\beta\gamma} + 2\partial_{\lambda}u^{\lambda}h^{\beta\gamma} - (u^{\alpha}\partial_{\lambda}h + \partial_{\lambda}u^{\lambda}h)\eta^{\beta\gamma} \\ &-2\partial_{\lambda}u^{\beta}h^{\gamma\lambda} - 2\partial_{\lambda}u^{\gamma}h^{\beta\gamma} + 2\partial_{\lambda}u_{\alpha}h^{\lambda\alpha}\eta^{\beta\gamma}]c_4\partial_{\beta}\Phi\partial_{\gamma}\Phi. \end{aligned} \tag{3.24}$$

Next the commutator

$$-4u^{\alpha}\partial_{\alpha}v^{\nu}[\partial_{\mu}v_{\nu}(x), \partial_{\beta}v^{\beta}(y)]2c_3\Phi^2$$

leads to

$$A_5 = -4ic_3u^{\alpha}\partial_{\alpha}v^{\nu}(x)\Phi^2(y)\partial_{\nu}^{y}\delta(x-y). \tag{3.25}$$

and finally

$$\begin{aligned} &-c_4u^{\beta}\partial_{\beta}\Phi\Big[\partial_{\mu}\Phi(x), \Phi^2(y)\left(mc_3h + 2c_3\partial_{\alpha}v^{\alpha} - \frac{c_4}{2}M^2h\right) \\ &\quad + c_4\partial_{\alpha}\Phi(y)\partial_{\nu}\Phi h^{\alpha\nu} + c_5\Phi^3(y)\Big] \end{aligned}$$

gives

$$\begin{aligned} A_6 = ic_4u^{\beta}\partial_{\beta}\Phi(x)\{&2\Phi\left(mc_3h - \frac{c_4}{2}M^2\right)h + 4c_3\Phi\partial_{\nu}v^{\nu} + 2c_4\partial_{\alpha}\Phi(y)h^{\alpha\nu}(y)\partial_{\nu}^{y} \\ &+3c_5\Phi^2\}\delta(x-y). \end{aligned} \tag{3.26}$$

The sum $A_1 + \cdots + A_6$ is equal to

$$\begin{aligned} B_1 = &-2ic_4(u^{\lambda}\partial_{\lambda}h^{\alpha\beta} + \partial_{\lambda}u^{\lambda}h^{\alpha\beta} - \partial_{\lambda}u^{\beta}h^{\alpha\lambda} - \partial_{\lambda}u^{\alpha}h^{\beta\lambda})\partial_{\alpha}\Phi\partial_{\beta}\Phi\delta \\ &+2ic_4^2u^{\beta}\partial_{\beta}\Phi(x)h^{\mu\nu}(y)\partial_{\mu}\Phi(y)\partial_{\nu}^{y}\delta(x-y) \quad (T1) \\ &-i(2mc_3 - M^2c_4)(u^{\mu}\partial_{\mu}h + \partial_{\mu}u^{\mu}h)\Phi^2\delta \\ &+ic_4(2mc_3 - M^2c_4)u^{\beta}\partial_{\beta}\Phi h\Phi\delta \quad (T2) \\ &+2i(2mc_3 - M^2c_4)\partial_{\nu}u_{\mu}h^{\mu\nu}\Phi^2\delta \quad (T3) \end{aligned}$$

$$-4ic_3u^\mu\partial_\mu v^\nu(x)\Phi^2(y)\partial^y_\nu\delta(x-y)+4ic_3c_4u^\beta\partial_\beta\Phi\Phi\partial_\mu v^\mu\delta \quad (T4)$$
$$+3ic_4c_5u^\beta\partial_\beta\Phi\Phi^2\delta. \quad (T5) \tag{3.27}$$

Following the methods developed in [9, Sect. 5.9] we have grouped the terms according to their type of Lorentz contractions. For example, $(T1)$ has $u^\lambda h^{\alpha\beta}\Phi\Phi$ and 3 derivatives which is different from $(T3)$. Only the terms within one type $T1,\ldots,T4$ can be combined to give a divergence. Due to the different coefficients c_4 and c_4^2 in T1 we must have $c_4=-2$ in order to get a divergence. If c_4 were $\neq -2$ then the last term of $(T1)$ would remain without compensation. Since this term is not a relative coboundary gauge invariance then would be violated.

The total anomaly is obtained by adding the contribution $x\leftrightarrow y$ according to (3.1). For the terms with $\delta(x-y)$ this simply gives factor 2. For the terms with derivative of δ we use the identity

$$g(x)f(y)\partial^y_\alpha\delta(x-y)+x\leftrightarrow y=(\partial_\alpha gf-g\partial_\alpha f)\delta(x-y) \tag{3.28}$$

Now the total anomalies of type T1 in (3.27) can be written in the form

$$(T1)_{\rm tot}=-4ic_4\big[(u^\lambda\partial_\lambda h^{\alpha\beta}+\partial_\lambda u^\lambda h^{\alpha\beta})\partial_\alpha\Phi\partial_\beta\Phi-\partial_\lambda u^\beta h^{\alpha\lambda}\partial_\alpha\Phi\partial_\beta\Phi$$
$$+u^\beta\partial_\alpha\partial_\beta\Phi\partial_\lambda\Phi h^{\alpha\lambda}-u^\beta\partial_\beta\Phi\partial_\alpha\Phi\partial_\lambda h^{\alpha\lambda}-u^\beta\partial_\beta\Phi\partial_\alpha\partial_\lambda\Phi h^{\alpha\lambda}\big]. \tag{3.29}$$

This agrees with the result in massless gravity [9, Eq. 5.9.40], and is a divergence

$$(T1)_{\rm tot}=-4ic_4\partial^x_\lambda\big[(u^\lambda h^{\alpha\beta}-u^\beta h^{\lambda\alpha})\partial_\alpha\Phi\partial_\beta\Phi\delta(x-y)\big]$$
$$+x\leftrightarrow y, \tag{3.30}$$

where $c_4=-2$ has been taken into account and will be assumed in the following. Type T2 is a divergence as well:

$$(T2)_{\rm tot}=-2i(M^2+mc_3)\partial_\mu[u^\mu h\Phi^2\delta(x-y)]+x\leftrightarrow y. \tag{3.31}$$

As in the massless case ([9, Eq. 5.9.45]) T3 is a coboundary:

$$(T3)_{\rm tot}=2(M^2+mc_3)d_Q[(h^2-2h_{\mu\nu}h^{\mu\nu})\Phi^2\delta(x-y)]. \tag{3.32}$$

Using the identity (3.28) we write T4 as follows

$$(T4)_{\rm tot}=-4ic_3[(\partial_\nu u^\mu+u^\mu\partial_\mu\partial_\nu v^\nu)\Phi^2-2u^\mu\partial_\mu v^\nu\Phi\partial_\nu\Phi$$
$$+4u^\mu\partial_\nu v^\nu\Phi\partial_\mu\Phi]\delta(x-y). \tag{3.33}$$

We first split off a divergence

$$(T4)_{\rm tot}=-4ic_3[2\partial_\mu(u^\mu\partial_\nu v^\nu\Phi^2)-\partial_\nu(u^\mu\partial_\mu v^\nu\Phi^2)$$
$$+2(\partial_\nu u^\mu\partial_\mu v^\nu-\partial_\mu u^\mu\partial_\nu v^\nu)\Phi^2]\delta(x-y). \tag{3.34}$$

Now the terms in the second line are a coboundary

$$(T4)_{\text{tot}} = -2ic_3[2\partial_\mu^x(u^\mu\partial_\nu v^\nu\Phi^2\delta) - \partial_\nu^x u^\mu\partial_\mu v^\nu\Phi^2\delta)] + x \leftrightarrow y$$
$$+8\frac{c_3}{m}d_Q[(\partial_\nu v^\mu\partial_\mu v^\nu - \partial_\mu v^\mu\partial_\nu v^\nu)\Phi^2\delta]. \tag{3.35}$$

Finally, T5 is a coboundary

$$T5 = \frac{12}{m}c_5 d_Q(v^\mu\Phi^2\partial_\mu\Phi\delta). \tag{3.36}$$

Adding the contribution $x \leftrightarrow y$ this gives the result of the theorem. □

4 Modified general relativity

As in ref. [8] we now consider the limit $m \to 0$ of vanishing graviton mass. The point is that this does not lead to massless gravity because the vector graviton field v^μ does not decouple from the other fields. In fact, in first order (proportional to Newton's constant) there survives the coupling term

$$T_v = h^{\mu\nu}\partial_\mu v^\lambda\partial_\nu v_\lambda. \tag{4.1}$$

If scalar matter is included then in addition to the standard coupling to the energy–momentum tensor of the scalar field ($\sim c_4$ in (3.13)) two further couplings T_3 and T_5 are possible. However, in second order the graviton mass appears in the denominator in N in (3.19). Consequently, if the coupling constants c_3 and c_5 do not depend on m, the limit $m \to 0$ exists for $c_3 = 0 = c_5$, only. Then we have no direct coupling of the v-field to normal matter; the resulting theory of [8] seems not to be physically relevant.

There is another option. Gauge invariance does not forbid the possibility that c_3 and c_5 depend on m, for example

$$c_3 = \lambda_3 m, \quad c_5 = \lambda_5 m, \tag{4.2}$$

where λ_j are independent of m. Then in the limit $m \to 0$ the first order trilinear couplings die away, but there remain the following quartic couplings from second order

$$T_{\Phi v} = 2\Phi^2\left\{8\lambda_3(\partial_\mu v^\mu\partial_\nu v^\nu - \partial_\mu v^\nu\partial_\nu v^\mu) - 12\lambda_5 v^\mu\partial_\mu\Phi\right\}. \tag{4.3}$$

In the classical limit this coupling must be added to the classical Lagrangean. As in the other coupling terms the usual factor $\sqrt{-g}$ is included. The necessity of this factor becomes clear when we derive the field equations below; but of course, an independent check by a third order calculation must be done. Our modification of general relativity is now defined by the following Lagrangean density

$$
\begin{aligned}
L_{\text{tot}} = & \frac{-2}{\kappa^2}\sqrt{-g}R + \sqrt{-g}g^{\mu\nu}\partial_\mu v_\lambda \partial_\nu v^\lambda \\
& + \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + \sqrt{-g}\Phi^2\left\{\lambda_3(\partial_\mu v^\mu \partial_\nu v^\nu - \partial_\mu v^\nu \partial_\nu v^\mu) + \lambda_5 v^\mu \partial_\mu \Phi\right\}\right. \\
& \left. - M^2\Phi^2\right). \qquad (4.4)
\end{aligned}
$$

The two terms in the first line are the pure gravitational interactions which have been studied already in [8]. The first term is the Einstein–Hilbert Lagrangean, $\kappa^2 = 32\pi G$ is essentially Newton's constant and R the scalar curvature. The second line contains the interaction with scalar matter; the numerical factors in (4.3) have been absorbed by redefining the coupling constants λ_3 and λ_5.

The Lagrangean (4.4) as it stands is Lorentz invariant, but the new terms in the second line are not invariant under general coordinate transformations. In [8] we have argued that this latter invariance can be maintained in the second term of the first line, if we consider v^λ as four scalar fields. This argument cannot be used for the new matter couplings in the second line. The lack of general covariance might be disturbing for classical relativists. However, one should keep in mind that classical general covariance corresponds to gauge invariance of the spin-2 quantum gauge theory, so that this principle is incorporated in the quantum theory. The latter is background dependent; we have selected Minkowski background. Returning again to the classical theory this background dependence remains and we get a Lorentz invariant classical theory, not a general covariant one. Still, by checking gauge invariance in third order we have to test whether there are further modifications in the classical theory. This will be done elsewhere.

The Euler–Lagrange equations for the Lagrangean (4.4) give the system of coupled field equations. Variation of $g^{\mu\nu}$ gives the modified Einstein equations

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{16\pi G}{c^3}\Big\{ & \partial_\mu v_\lambda \partial_\nu v^\lambda - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha v_\lambda \partial_\beta v^\lambda \\
& + \frac{1}{2}\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{4}g_{\mu\nu}(g^{\alpha\beta}\partial_\alpha \Phi \partial_\beta \Phi - M^2\Phi^2) \\
& - \frac{1}{2}g_{\mu\nu}\Phi^2\left[\lambda_3(\partial_\alpha v^\beta \partial_\beta v^\alpha - \partial_\alpha v^\alpha \partial_\beta v^\beta) + \lambda_5 v^\alpha \partial_\alpha \Phi\right]\Big\}. \qquad (4.5)
\end{aligned}
$$

The variational derivative with respect to v^μ yields

$$
\begin{aligned}
2\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta v_\mu) = & -2\lambda_3\left[\partial_\mu(\sqrt{-g}\Phi^2\partial_\nu v^\nu) - \partial_\nu(\sqrt{-g}\Phi^2\partial_\mu v^\nu)\right] \\
& + \lambda_5\sqrt{-g}\Phi^2\partial_\mu \Phi. \qquad (4.6)
\end{aligned}
$$

Here the vector-graviton field has source terms from the new scalar–matter coupling. Note that the second order derivative $\partial_\mu \partial_\nu v^\nu$ cancels on the right-hand side so that we have a wave equation with source. Finally, the variation of Φ gives the Klein–Gordon equation in the metric $g^{\alpha\beta}$ plus source terms:

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\Phi) + M^2\Phi = 2\lambda_3\Phi(\partial_\mu v^\mu\partial_\nu v^\nu) - \partial_\nu v^\mu\partial_\mu v^\nu)$$
$$-\lambda_5\Phi^2\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}v^\mu). \tag{4.7}$$

The investigation of the modified coupled Einstein-, wave and Klein–Gordon Eqs. (4.5–7) is a complicated problem. Some qualitative conclusions can be drawn already. If the coupling constants λ_3 and λ_5 in (4.6) are small then the vector graviton field v_μ must be also small. For $v_\mu \to 0$ (4.5) goes over into the ordinary Einstein equations. Therefore, for small enough λ_3, λ_5 the theory certainly passes the classical solar system tests of general relativity. It remains to be investigated whether on the galactic scale the additional contributions of the v-field to the energy–momentum tensor in (4.5) give an explanation of the dark matter phenomenology.

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References

1. Dragon, N.: BRS symmetry and cohomology, Schladming lectures. hep-th/9602163
2. Grigore, D.R.: On the quantization of the linearized gravitational field. Class. Quant. Gravit. **17**, 319 (2000). hep-th/9905190
3. Grigore, D.R.: Cohomological aspects of gauge invariance in the causal approach. Rom. J. Phys. **55**, 386–438 (2010). hep-th/0711.3986
4. Grigore, D.R.: Perturbative gravity in the causal approach. Class. Quant. Gravity **27**, 0150013 (2010). hep-th/0805.3438
5. Grigore, D.R., Scharf, G.: Massive Yang–Mills fields in interaction with gravity. hep-th/0808.3444
6. Grigore, D.R., Scharf, G.: Massive gravity as a quantum gauge theory. Gen. Relativ. Gravit. **37**(6), 1075 (2005). hep-th/0404157
7. 't Hooft, G.: arXiv:0708.3184 v4 (7 Jan 2008)
8. Scharf, G.: From massive gravity to modified general relativity. Gen. Relativ. Gravit. **42**, 471–487 (2010). doi:10.1007/s10714-009-0864-0
9. Scharf, G.: Quantum Gauge Theories—Spin one and Two. Google books (2010)